CS 591 K1:
Data Stream Processing and Analytics

Spring 2020

4/23: Cardinality and frequency estimation

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Counting distinct elements

How can we count the number of **distinct elements** seen so far in a stream?

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Naive solution: maintain a hash table

Convert the stream into a multi-set of *uniformly distributed* random numbers using a *hash function*.

The more different elements we encounter in the stream, the more different hash values we shall see.

Let **h** be a hash function that maps each stream element into $M = log_2 N$ bits, where N is the domain of input elements:

$$h(x) = \sum_{k=0}^{M-1} i_k 2^k = (i_0 i_1 \ldots i_{M-1})_2, i_k \in \{0,1\}$$

For each element x, let $rank(x)$ be the number of *0s* in the end of $h(x)$:

- e.g.
  - x_1 = 318, $h(x_1)$ = 12 or 011**00** => $rank(x_1)$ = 2
  - x_2 = 9013, $h(x_2)$ = 24 or 11**000** => $rank(x_2)$ = 3

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Claim: The maximum observed rank is a good estimate of $\log_2 \mathbf{n}$.

In other words, the estimated number of distinct elements is equal to:

$$\hat{n} = 2^R$$

The hash function h hashes x to any of N values with probability $1/N$.

Out of all x we hash:

- around 50% will have a binary representation that ends in at least one 0:
  - ********0 (the probability of a 0 is 1/2)
- around 25% will end in at least two 0s:
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- and so on…

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It takes 2^r hash calls before we encounter a result with r 0s.

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The probability that a given $h(x)$ ends in at least r 0s is:

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$$(1 - 2^{-r})^k$$

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The estimate 2^R cannot be too high or too low.

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- R = 4, 2^R = 16 distinct elements
- R = 5, 2^R = 32 distinct elements
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To get a better estimate, we need to use **multiple hash functions** and combine their estimates:

- Using many hash functions for a high-rate stream is expensive
- Finding many random and independent hash functions is difficult

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We split the input stream into $m = 2^p$ sub-streams $S_0, S_1, \ldots, S_{m-1}$

For every element x, we compute $h(x)$ and use the p first bits of the M-bit hash value to select a sub-stream and the next M-p bits to compute the $rank(.)$:

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For $h(x) = (i_0 i_1 \ldots i_{M-1})_2, i_k \in \{0,1\}$ we select one of m counters

COUNT[j], where $j = (i_0 i_1 \ldots i_{p-1})_2$

Stochastic averaging: example

Let $M = 5$, $p = 2$ and a hash function h_5 that maps elements to a binary representation of length 5.

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Consider the input elements {5, 14, 5, 2, 8, 1, …}

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LogLog algorithm

```
Input: stream S, array of m counters, hash fiction h
Output: cardinality of S

for j=0 to m-1 do:
 COUNT[j] = 0

for x in S do:
  i = h(x)
  j = getLeftBits(i, p)
  r = rank(getRightBits(i, M-p))
  COUNT[j] = max(COUNT[j], r)

  R = average(COUNT) // average of all j counters
  output a * m * 2R // a is a constant, a ≈ 0.39701, for m ≥ 64.
```

Why *LogLog*?

Let's assume we want to be able to count up to $\mathbf{n}$ distinct elements.

We need a hash function that maps each input element to $\mathbf{log_2 n}$ bits.

Then, each counter needs to be able to count up to $\mathbf{log_2(log_2 n)}$ 0s.

Combining estimates

- **Average won't work**: The expected value of 2^R is too large.
- **Median won't work**: it is always a power of 2, thus, if the correct estimate is between two powers of 2, we won't get a good estimate.

Solution: **harmonic mean** (HyperLogLog)

$$\hat{n} = a_m \cdot m^2 \cdot (\sum_{j=0}^{m-1} 2^{-COUNT[j]})$$

Standard error

The standard error of the LogLog algorithm is inversely related to the number of counters m:

$$\delta \approx \frac{1.3}{\sqrt{m}}$$

For $m = 256$, the error is about 8%

For $m = 1024$, the error decreases to 4%

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- Each counter needs to be able to count up to 20 0s, so we need to allocate $\log_2 20$ = 4.32 bits per counter.
- If we round up to 5 bits, that's **640 bytes in total**.

Estimating frequencies

Motivating examples

Detect DNS DDoS attacks

- Flooding the resources of the targeted system by sending a large number of query from a botnet
- Group queries by their top-level domain and investigate most popular domains
- Alert if we detect many different non-existent subdomains of the same primary domain

Trending topics calculation

- Twitter receives around 500 billion tweets per day
- Estimating the frequencies of hashtags and comparing them with yesterday's frequencies provides an indication of what is "trending"

Counting Bloom Filter

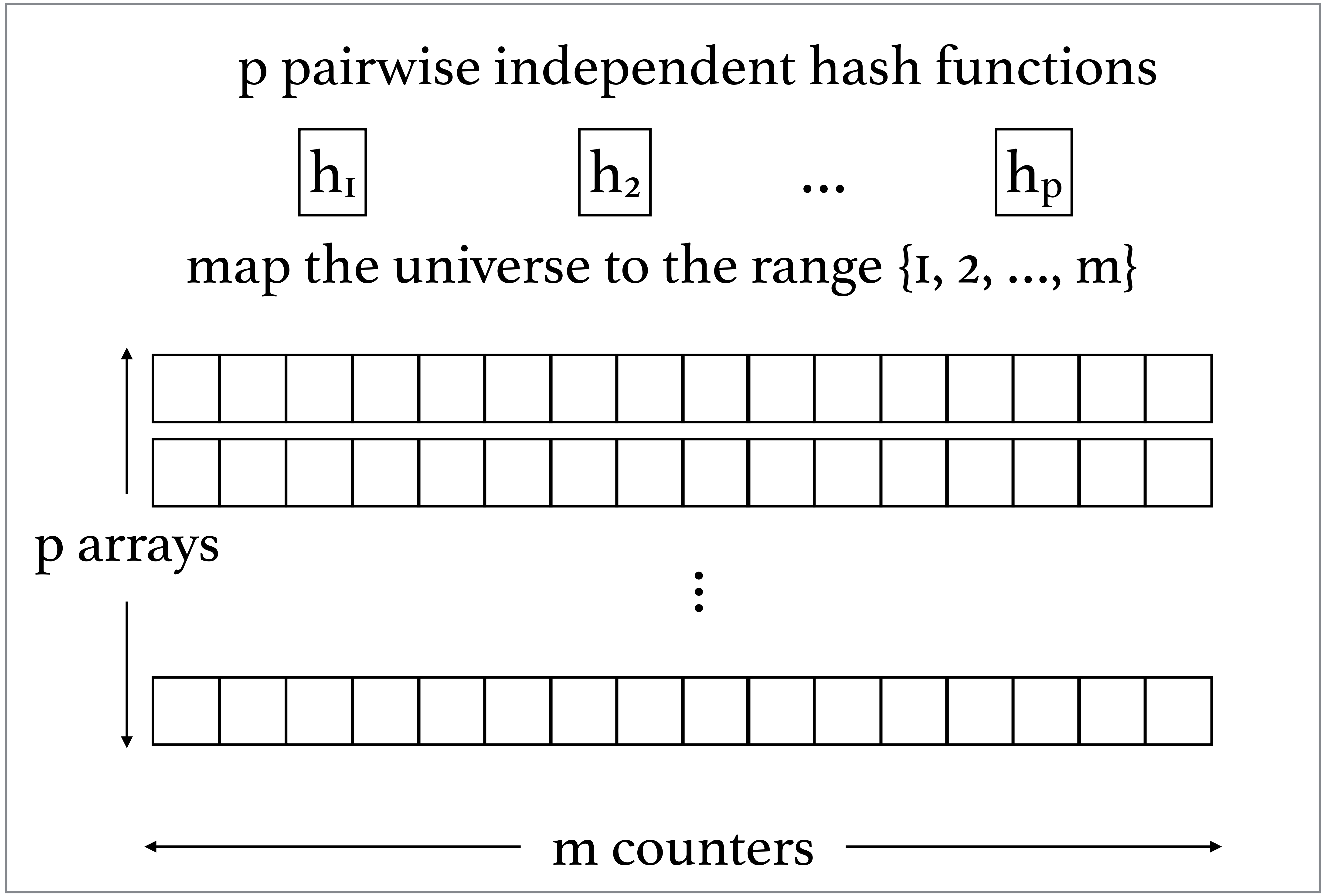
- Expand the classical BF with an array of $\mathbf{m}$ counters corresponding to each of the $\mathbf{m}$ bits in the filter:
  - Increment the corresponding counter every time an element is added
  - To delete an element, decrease its corresponding counters and unset the corresponding bit of the counter falls to 0
- A single array of counters for all hash functions increases the collision probability
- Counter overestimation is almost certain for very large data streams with high-frequency elements

The Count-Min Sketch

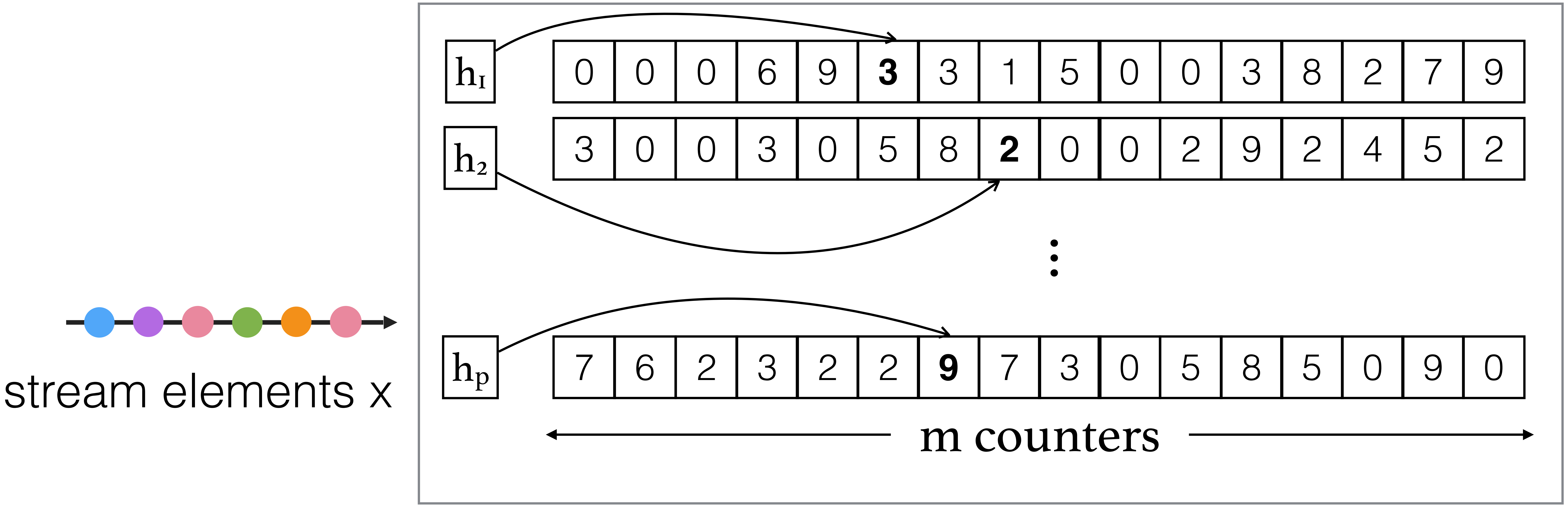
- A space-efficient probabilistic data structure that can be used to estimate frequencies and heavy hitters in data streams
- It was introduced in 2003 by Cormode and Muthukrishnan

- It uses a hash table of $\mathbf{p}$ arrays of $\mathbf{m}$ counters
- Elements update different subsets of counters, one per hash table
- Many independent trials by using $\mathbf{p}$ hash functions with an array of $\mathbf{m}$ counters for each of them

The Count-Min Sketch



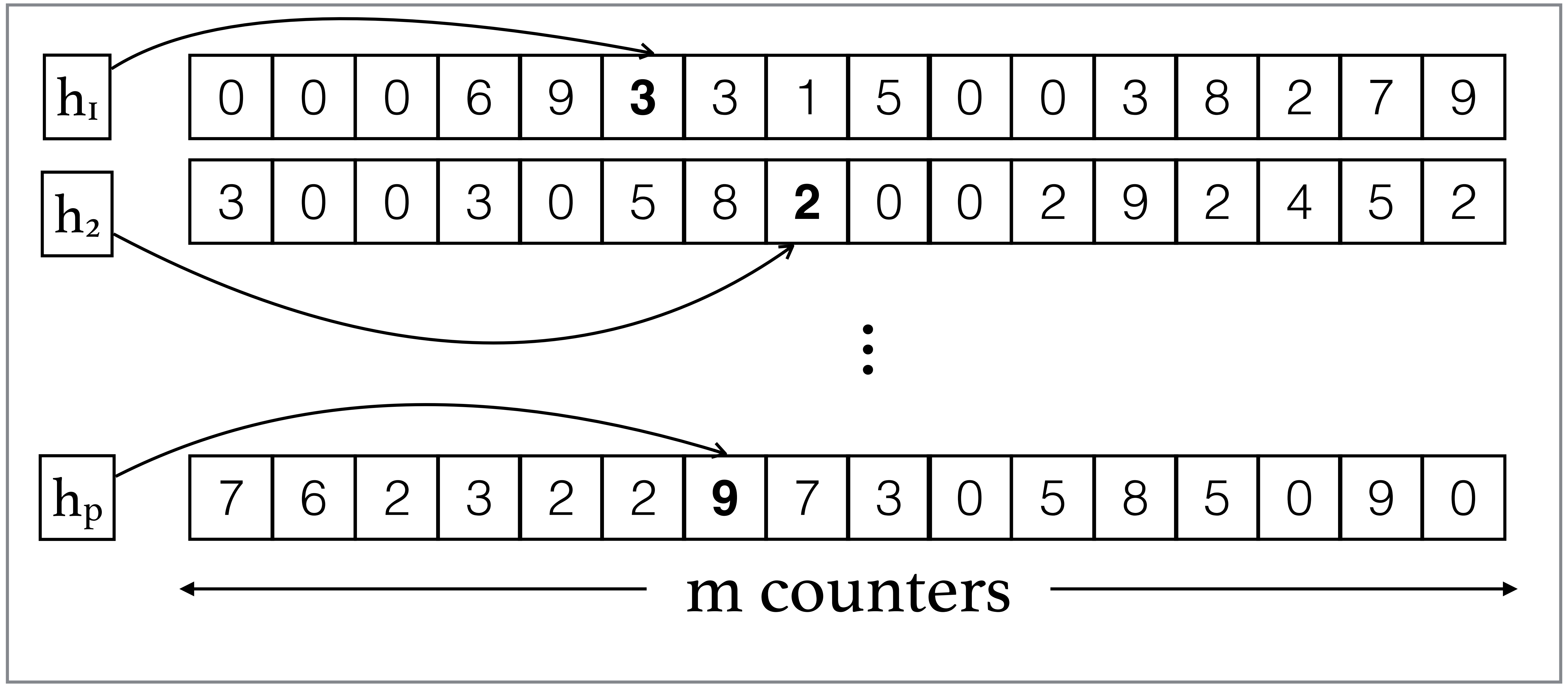
Adding an element to the sketch



All counters are initialized to 0s

```
for j=1 to p do
  i = hj(x)
  ci,j++
```

Estimating frequency




```
let f: array of length p
for j=1 to p do
  i = hj(x)
  f[j] = ci,j
return min(f[1], f[2], …, f[p])
```

Counters provide the upper bound for an element's frequency:

$$f(x) \leq c_j^{h(x)}, j = 1,2,\ldots,p$$

Because $\mathbf{m} << \mathbf{n}$, there are many collisions and counters generally overestimate real frequencies.

The best approximation is not the average of all counters, but the **minimum**.

Computing top-k

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- When a popular element's frequency drops below the threshold, we remove it from the heap

Computing top-k

```
N=0 // number of elements so far
X* = {} // heap of top-k elements

for x in input do:
  N = N+1
  f* = N/k // current frequency threshold
  update(x) // add x to the count-min sketch (slide 22)
  f = frequency(x) // use sketch to estimate frequency (slide 23)

  if f >= f* then:
    X*.add({x, f})
  // remove unpopular elements from the heap
  for (y, f_y) in X* do:
    if f_y <= f* then
      X*.remove({y, f_y})

return X*
```

Error and space/time trade-offs

- Query approximation error ϵ
- Error probability δ

Guarantee: The estimation error for frequencies will not exceed $\epsilon \cdot n$ with probability $1 - \delta$

- A higher number of hash functions decreases the probability of a bad estimate: $p = \lceil ln\frac{1}{\delta} \rceil$
- The recommended number of counters is $m = \lceil \frac{2.71828}{\epsilon} \rceil$

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Considering 32-bit counters, the count-min sketch requires a total of **54.4MB** of memory.

Further reading

- Jure Lescovec, Anand Rajaraman and Jeffrey David Ullman. **Mining of Massive Datasets**. http://infolab.stanford.edu/~ullman/mmds/book.pdf
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